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## Can we live in the bulk without a brane?

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### ABSTRACT

We suggest a braneless scenario that still hides large-volume extra dimensions. Ordinarily the strength of bulk gauge interactions would be diluted over the large internal volume, making all the four-dimensional forces weak. We use the fact that if the gauge fields result from the dimensional reduction of pure higher-dimensional gravity, then the strengths of the four-dimensional gauge interactions are related to the sizes of corresponding cycles averaged over the compact internal manifold. Therefore, if a gauge force is concentrated over a small cycle it will not be diluted over the entire manifold. Gravity, however, remains diluted over the large volume. Thus large-volume, large mass-gap extra dimensions with small cycles can remain hidden and result in a hierarchy between gravity and the other forces. However, problematically, the cycles are required to be smaller than the higher-dimensional Planck length and this raises concern over quantum gravity corrections. We speculate on possible cures.

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A low-budget yet concrete observation about our universe is that it is four-dimensional, and although human beings by the billions confirm this observation daily, it might not be true. Our universe may have multiple extra spatial dimensions and only *appear* to be four-dimensional. Extra dimensions could hide if they are curled up so small that no observations to date could excite modes energetic enough to probe these directions. Originally, interest in large-volume extra dimensions inspired braneworlds as a means to hide the extra dimensions — float our universe on a 3-brane and confine all Standard Model fields to that braneworld [1–3]. In this Letter, we describe a braneless alternative that allows us to hide large-volume extra dimensions. The internal manifolds have in common three essential features: (1) large volume, (2) lowest modes that are energetically expensive despite the large volume, and as we will see (3) some small Killing cycles. However, the small cycles are very small and therefore penetrate quantum gravity scales as we discuss shortly.

We can lend intuition for why these three features are essential. Part of the picture was presented in a previous article [4]. It is commonly assumed that the larger the volume, the lower the natural harmonics on the space. The lower the notes, the less the effort that is required to play them. If we were not confined to a brane, we should expect to have observed a large internal volume

already. However, this expectation is contradicted by an infinite number of known manifolds that despite their large volume have no low notes [4–8]. Such spaces would remain hidden despite their large size because it remains too energetically expensive to probe them.

Still, an expensive spectrum of modes is not sufficient to free the world from incarceration on a brane. If Standard Model fields were allowed to live in the bulk, it might seem that the strength of all the gauge interactions would be diluted over the large volume — just as the strength of gravity would be diluted — and all forces would be weakened relative to the true fundamental scale. For this reason, Standard Model gauge fields were localized on a brane while the gravitational field inhabited the bulk. This split between habitats enforced a hierarchy between gravity and particle physics [1–3].

If we are to do away with the brane altogether and allow all fields to fill the bulk, then we need to save the gauge couplings from dilution over the large internal volume. We show here that this is possible if the gauge fields are generated by the dimensional reduction of a purely gravitational field — as in the original Kaluza–Klein reduction [9,10] — and there are some small Killing cycles around the internal volume. In this picture, a photon is really a metric oscillation around an  $S^1$ , and the strength of its coupling is inversely proportional to the size of that cycle. So while the volume is large, electromagnetic interactions involve only one small circle and not the entire manifold. The dilution over the internal volume is compensated by localizing all gauge interactions over small cycles, instead of confining them to a brane.

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We will review the dimensional reduction of gravitational fields from a higher-dimensional universe down to a 4-d universe and show that the conditions we want our manifold to satisfy are the following:

*Large volume:* Under dimensional reduction, the integrated volume  $\mathcal{V}$  (in units of  $M^N$ ) is related to the ratio of the observed 4-d Planck mass,  $m_p$ , to the fundamental higher-dimensional Planck scale,  $M \sim \text{TeV}$ , through

$$\mathcal{V} = \left( \frac{m_p}{M} \right)^2. \quad (1)$$

This only generates a hierarchy if the Higgs mass is small compared to  $m_p$ . Since the 4-d metric is not warped, any bulk scalar field added by hand will automatically have the mass it did in the bulk [1,2]. Although vacuum expectation values and couplings will be affected, combinations of them lead to invariant masses [4]. So if the Higgs is a bulk scalar field of mass  $M$  in the bulk, it will reduce to a 4-d scalar field of mass  $M$ .

*Large mass gap:* The mass gap, set by the minimum non-zero eigenvalue of the Laplacian on the internal manifold, must be large to suppress Kaluza–Klein excitations, a condition we express as,

$$m_{KK} \gtrsim M. \quad (2)$$

*Small cycles:* Some of the 4-d gauge couplings, i.e. those of the Standard Model, must be of order one. The 4-d gauge couplings can be expressed as

$$g_{4d} \sim (\langle s^2 \rangle^{1/2} M \mathcal{V}^{1/2})^{-1}, \quad (3)$$

where  $\langle s^2 \rangle^{1/2}$  is the root mean square of the circumference of the corresponding Killing cycle over the internal space (as we review below [11]). Setting this  $\simeq 1$  gives

$$\langle s^2 \rangle^{1/2} \sim (M \mathcal{V}^{1/2})^{-1}. \quad (4)$$

When all three conditions are met, we have large-volume extra dimensions that can be hidden without invoking a brane while still affecting a hierarchy between the weakness of gravity relative to particle interactions.

However, problematically, by Eq. (1) in Eq. (4),

$$\langle s^2 \rangle^{1/2} \sim m_p^{-1}. \quad (5)$$

The cycles corresponding to gauge interactions are smaller by a factor of the hierarchy than the higher-dimensional Planck length, leading to curvature invariants (or analogous topological invariants) that are large and susceptible to uncontrolled quantum corrections. We will consider internal manifolds that are a direct product of submanifolds as well as internal manifolds that are warped products of submanifolds. Although the internally warped spaces seem promising in that the small cycles are of order  $M^{-1}$ , the warping shrinks the cycles over the span of the manifold so they are metrically small in places. Consequently, we run into trouble with large curvature invariants and cannot claim controlled quantum gravity corrections.

## 1. Gravity reduction

The gravity reduction of Kaluza and Klein [9,10] provided a remarkable, explicit demonstration of unification: a metric flux around a circle in 5-d appeared to the 4-d world as the photon. Since then, all manner of gauge groups have been shown to result from the dimensional reduction of pure gravity over higher-dimensional manifolds. We consider pure  $(4+N)$ -d gravity

$$S = \frac{M^{2+N}}{2} \int d^{4+N}x \sqrt{-G} R(G), \quad (6)$$

on a product space  $\mathcal{M} \times \mathcal{N}$ , where  $\mathcal{M}$  is 4-d and the internal  $N$ -d manifold  $\mathcal{N}$  has isometry group  $\mathcal{G}$ . Let us consider only zero-modes under dimensional reduction, which is equivalent to assuming that the Kaluza–Klein excitations of the metric can be ignored at the energy scales we are considering. Then we use the ansatz for the metric [11,12]

$$G_{AB} = \left( \begin{array}{c|c} g_{\mu\nu} + A_\mu^j A_\nu^i \xi_i^m \xi_j^n \hat{g}_{mn} & A_\mu^i \xi_{in} \\ \hline A_\nu^j \xi_{jm} & \hat{g}_{mn} \end{array} \right), \quad (7)$$

where  $\mu = 0, \dots, 3$  runs over 4-d coordinates  $x$  and  $m = 5, \dots, 4+N$  runs over the internal coordinates  $y$ . The  $\xi_i^m(y)$  are the Killing vectors of  $\mathcal{N}$  that under the Lie bracket obey the algebra of the isometry group of the internal manifold,

$$[\xi_i, \xi_j]^\mu = C_{ij}^k \xi_k^\mu, \quad (8)$$

with  $C_{ij}^k$  the canonical ( $\sim 1$ ) structure constants of the algebra. In words, the internal spacetime symmetries appear to us in 4-d to be proper gauge symmetries, and the off-diagonal excitations of the metric camouflage as gauge fields.

After dimensional reduction, we get 4-d gravity with metric  $g_{\mu\nu}$ , 4-d Yang–Mills gauge fields  $A_\mu^i$  with gauge group isomorphic to  $\mathcal{G}$ , and scalar field moduli from the internal metric  $\hat{g}_{mn}$ ,

$$\int d^{4+N}x \sqrt{-G} \frac{M^{2+N}}{2} \left[ R(G) - \hat{g}_{mn} \xi_i^m \xi_j^n \frac{1}{4} F_{\mu\nu}^i F^{j\mu\nu} \right] + \dots \quad (9)$$

where  $F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + C_{jk}^i A_\mu^j A_\nu^k$  is the standard non-abelian Yang–Mills curvature. The  $\dots$  indicates additional moduli terms, including curvature terms like  $R(\hat{g})$  that serve as a potential for the moduli of the internal dimension and/or contribute to the cosmological constant. (While both moduli stabilization and the cosmological constant are important problems for any higher-dimensional model, we defer to the rich literature on the subjects.)

Integrating the Einstein–Hilbert term over  $y$ , we see that the  $(4+N)$ -d Planck constant is related to 4-d Planck constant by

$$\mathcal{V} \equiv M^N \int d^N y \sqrt{\hat{g}} = \left( \frac{m_p}{M} \right)^2. \quad (10)$$

This is the first condition, Eq. (1). The internal volume under dimensional reduction must be large relative to the fundamental scale  $M$ .

For a given simple part of the gauge group, the kinetic coefficient of the gauge fields can be chosen diagonal, and its coefficient will be related to the 4-d gauge coupling  $g_{4d}$ ,

$$\frac{M^2 \mathcal{V}}{2} \langle \hat{g}_{mn} \xi_i^m \xi_j^n \rangle = \frac{1}{g_{4d}^2} \delta_{ij}, \quad (11)$$

where  $\langle \varphi(y) \rangle = M^N \mathcal{V}^{-1} \int d^N y \sqrt{\hat{g}} \varphi(y)$  indicates an average of a function  $\varphi(y)$  over the internal volume.

The action is then the canonical action for 4-d gravity with 4-d gauge fields:

$$S = \int d^4x \sqrt{-g} \left[ \frac{m_p^2}{2} R(g) - \frac{1}{4g_{4d}^2} (F_{\mu\nu}^i)^2 \right] + \dots \quad (12)$$

The massive gravitons corresponding to Kaluza–Klein modes for the metric that would appear in the action have masses  $\sim m_k^2$  corresponding to eigenvalues of the scalar Laplacian on the higher-dimensional manifold,

$$\nabla_{(N)}^2 \psi_k = -m_k^2 \psi_k. \quad (13)$$

To suppress Kaluza–Klein modes, we require non-zero eigenvalues of the Laplacian to be large,

$$m_{k_{\min}} \gtrsim M. \quad (14)$$

This is the second condition, Eq. (2).

To obtain the third condition, Eq. (4), we give a minimal review of the argument in [11], to show that gauge coupling constants in the lower-dimensional theory, given by Eq. (11), can be interpreted as averaged circumferences over the compact internal manifold. Given a compact, simple Lie group acting on a compact manifold, there is a Killing vector  $\xi^m$  corresponding to each Lie algebra generator  $T_i$ . A generic Killing vector  $\xi^m$  generates closed orbits  $Y^m(\lambda)$ , parametrized by some  $\lambda$ , in the corresponding compact manifold,

$$\frac{d}{d\lambda} Y^m(\lambda) = \xi^m(Y(\lambda)). \quad (15)$$

Given a starting point  $y$  for the orbit, the solution is

$$Y^m(\lambda, y) = e^{\lambda \xi^n(y) \partial_n} Y^m(0, y), \quad (16)$$

where the partial derivative in the exponent is with respect to  $y$ . This generates the exponential map on the group manifold, and since the generators are normalized canonically (structure constants are  $\sim 1$ ), and the group is compact, the curve comes back to its starting point after some order one range of  $\lambda$  (usually  $\lambda_{\max} = 2\pi$ ).

The circumference of this curve is thus

$$s(y) = \int_0^{\lambda_{\max}} d\lambda \sqrt{\hat{g}_{mn}(Y) \frac{dY^m}{d\lambda} \frac{dY^n}{d\lambda}}. \quad (17)$$

By differentiating the quantity in the square root with respect to  $\lambda$ , using Eq. (15), and using Killing's equation  $\mathcal{L}_\xi \hat{g}_{\mu\nu} = 0$ , it is straightforward to check that the integrand is actually independent of  $\lambda$ , so we have

$$s(y) \sim \sqrt{\hat{g}_{mn} \xi^m \xi^n}. \quad (18)$$

Taking the average as  $y$  varies over the submanifold  $\mathcal{N}$ , we have from Eq. (11)

$$g \sim \frac{1}{M \mathcal{V}^{1/2} \sqrt{\langle s^2 \rangle}}. \quad (19)$$

The argument also generalizes to the case of a  $U(1)$  group factor, though one has to couple in matter to read off the coupling strength [11]. Weak or strong gauge couplings correspond to large or small cycles respectively.

## 2. Direct product spaces: No warping

To emphasize, we are promoting the new observation that a hierarchy between gauge and gravitational couplings could come from the geometry of internal manifolds alone, using a traditional Kaluza–Klein reduction and allowing all fields to live in the bulk. Construction of braneless models relies heavily on creating gauge fields from higher-dimensional gravity via the mechanism that we discussed at the end of the previous section. We combine that mechanism with geometries that satisfy the three criteria we set out in the introduction to illustrate our proposal through several examples.

Note that in our examples, the Einstein equations can always be satisfied with an appropriate choice of matter densities in the

extra dimensions. However, since our primary objective here is to achieve a hierarchy from the shape of extra dimensions – an objective to be contrasted against achieving a hierarchy by confining some fields to a brane – exact solutions to Einstein's equations are currently outside the scope of this Letter.

Consider as a first example a direct product of 4-d  $\mathcal{M}$  with an  $N$ -dimensional manifold built from a string  $S^n \times \cdots \times S^n$ , of  $D$  hyperspheres of radius  $R_1$  [4] taken in product with one much smaller  $n$ -sphere of radius  $R_2$ , so  $N = n(D+1)$ . The gauge fields will come from the smaller sphere, and we have  $\langle s^2 \rangle^{1/2} \sim R_2$ . By our 3 conditions, the internal space is subject to the constraints:

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_{S_1}^D \mathcal{V}_{S_2} \sim (R_1 M)^{Dn} (R_2 M)^n \sim (m_p/M)^2, \\ m_{KK} &\sim R_1^{-1} \sim M, \\ g_2 &\sim (R_2 M \mathcal{V}^{1/2})^{-1} \sim (R_2 m_p)^{-1} \sim 1. \end{aligned} \quad (20)$$

Choosing  $R_2 \sim m_p^{-1}$ ,  $R_1 \gtrsim M^{-1}$  and  $D \gg 1$  easily satisfies all 3 conditions. The gauge fields from  $R_2$  couple with strength  $g_2 \sim 1$  while the string of large  $S^n$ 's will couple with a strength suppressed by a factor of  $M/m_p \sim 10^{-16}$ .

In general we can build the internal manifold as a product of any large-volume, large mass gap space with highly diluted gauge couplings times a small manifold with undiluted gauge couplings. Another interesting internal manifold is provided by a squashed  $T^2$  in product with a small space. Unlike the previous example, this manifold does not require large dimensionality. The large volume, large mass gap comes from the squashed  $T^2$  as was shown in [13, 14] while the undiluted gauge coupling could come from a small internal manifold such as  $\mathbb{CP}^2 \times \mathbb{S}^2 \times \mathbb{S}^1$ , which has the isometries of the Standard Model gauge group [15].

There are an infinite number of 2-surfaces that could participate in this construction. In [4], compact hyperbolic 2-surfaces were considered. Surfaces of arbitrarily large genus, and therefore arbitrarily large area,  $A = 4\pi(g-1)R^2$ , were shown to have minimum eigenvalue of roughly  $k_{\min} \sim 1/(2R)$  [16–21]. With  $R \sim M^{-1}$  and  $g \gg 1$ , these qualify as large mass-gap, large-volume manifolds. Additionally, hyperbolic spaces have no Killing vectors and so would not contribute any additional, unwanted gauge fields. We could equally well take the internal space to be a product of these large-volume, large mass-gap manifolds with a small internal manifold whose isometries generate the Standard Model.

These direct product spaces have a nice interpretation: There is a large internal volume but gauge fields correspond to excitations along small cycles and so do not require ringing the whole big manifold, just a small piece of it. Therefore the gauge coupling is not diluted, while gravity is.

Despite this nice interpretation, these examples are flawed. One of the geometric scales,  $m_p^{-1}$ , is many orders of magnitude smaller than the fundamental length scale,  $M^{-1}$ , with the considerable disadvantage that small cycles might force us into quantum gravity arenas, undermining the consistency of the analysis. The concern is that higher-dimensional operators of the form  $R^2$ ,  $R_{\mu\nu} R^{\mu\nu} \dots$  become significant. Before we speculate on possible resolutions, we turn to warped internal spaces next, although we will see that these also have the problem of small cycles.

## 3. Warped internal spaces

Consider now a product of  $\mathcal{M}$  with an internal space  $\mathcal{N} = H^2 \times \mathcal{N}_G$  that is a product of a swath of the hyperbolic plane,  $H^2$ , with coordinates  $y, z$ , and a space  $\mathcal{N}_G$  with coordinates  $\hat{x}^m$  and metric  $\hat{g}_{mn}$  with isometry group the desired gauge group. Let  $\mathcal{N}_G$  carry a warp factor  $f(y)$  dependent on only the  $y$  coordinate

of the hyperbolic plane (note that the warping is internal and that the 4-d metric carries no warp factor),

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + f(y) \hat{g}_{mn} d\hat{x}^m d\hat{x}^n + \frac{L^2}{y^2} (dy^2 + dz^2). \quad (21)$$

We have used the upper half-plane representation of the hyperbolic plane, with  $L$  the curvature scale. We take a square swath on the plane between the limits  $0 < z < 1$  and  $\epsilon < y < 1$  where  $\epsilon$  is small but non-zero in order to render lengths and areas finite.

In order for all components of the higher-dimensional metric to transform properly under the gauge transformations and create the illusion of gauge bosons, we require the Killing vectors that generate them to be Killing vectors of the *entire* metric, not just of the submanifold  $\mathcal{N}_G$ . But because the warp factor depends only on the coordinates of  $H^2$ , Killing vectors of  $\mathcal{N}_G$  are automatically Killing vectors of the entire internal manifold. Finite volume hyperbolic manifolds do not have Killing vectors and so do not introduce additional gauge fields.

Choosing  $f(y) = \alpha y$ , with  $\alpha$  some order one constant, and a submanifold  $\mathcal{N}_G$  of dimension 2 and un-warped volume  $\mathcal{V}_{\mathcal{N}_G} \sim 1$  in units of  $M^2$ , the 4-d Planck mass by Eq. (10) is

$$\begin{aligned} \frac{m_p^2}{M^2} &= \mathcal{V}_{\mathcal{N}_G} M^2 L^2 \int_0^1 dz \int_\epsilon^1 dy y^{-2} f(y) \\ &= \alpha \mathcal{V}_{\mathcal{N}_G} (LM)^2 \ln(1/\epsilon). \end{aligned} \quad (22)$$

The 4-d Planck mass is then large if we take  $\epsilon$  very near zero. We thereby meet condition Eq. (1) to create a weakened strength of gravity compared to a fundamental scale.

To check the mass gap of Eq. (4), we note that the KK modes on  $\mathcal{N}_G$  should be of order  $M^{-1}$ . Also, the eigenmodes on the swath of the hyperbolic plane should be subcurvature modes and therefore the eigenvalues are bounded from below by  $1/(2L)$  so the mass gap is comfortably large for  $L \sim M^{-1}$ .

The gauge couplings are set by the normalization of the Killing vectors, Eq. (11). Due to the factor of  $\hat{g}_{mn}$  in that inner product, an additional factor of  $f$  is introduced so that the gauge couplings consistent with Eq. (19) are set by  $\mathcal{V}\langle s^2 \rangle = \mathcal{V}_{\mathcal{N}_G} R^2 M^2 L^2 \int (f^2/y^2) dz dy = \alpha^2 \mathcal{V}_{\mathcal{N}_G} (LM)^2 R^2 (1 - \epsilon)$  where  $R$  is the characteristic size of the un-warped cycle on the submanifold  $\mathcal{N}_G$ . It follows that

$$g \sim (\alpha M^2 L R \mathcal{V}_{\mathcal{N}_G}^{1/2})^{-1}. \quad (23)$$

For  $\mathcal{V}_{\mathcal{N}_G} \sim LM \sim RM \sim \alpha \sim 1$ , we have an  $\mathcal{O}(1)$  coupling.

The hierarchy between fundamental scales has been shifted to a hierarchy between geometric scales. As with the direct product, the warped internal product dilutes gravity over a large internal volume while gauge fields correspond to excitations over small cycles. Unfortunately, also like the direct product, the cycles vary by  $\sqrt{f(y)}R$  and are metrically small in places. The hazards of quantum gravity thus reappear as the bulk Ricci scalar on the internal space is larger by the warp factor than is tolerable, with a contribution of the form  $f^{-1}R(\hat{g})$ .

#### 4. Speculation

It may be that there is a no-go theorem that ensures the small cycles we need are always catastrophically small if there are no branes. On the other hand, although we did not present the calculation here, we find the dimensional reduction of pure gravity where the external 4-d space is warped as in Randall–Sundrum – as opposed to the internal warping detailed above – does allow for gauge fields to live in the bulk with order unity

gauge couplings following this prescription. However, this construction is less novel, and furthermore, the hierarchy requires the Higgs to be constrained to a brane so does not qualify as braneless.

Finally, it is a celebrated result of string theory that small cycles are controlled in the UV theory as extra light degrees of freedom resurface and naturally resolve any metrical divergences [22]. We speculate that in a stringy formulation, a similar resolution may allow a fully braneless model with small cycles and good gauge couplings. If string theory is the UV completion, there is also the worry that winding modes around a homotopically non-trivial small cycle can become light, since their mass  $\sim R/\alpha'$ . This is not a problem if the Killing cycles are homotopically trivial, as in our example with spheres  $S^n$ . We also mention that there are explicit examples of hyperbolic 3-folds with large volume and small geodesics (see Snappea [23]). The curvature invariants could be stabilized at  $M^{-1}$  while sustaining large volume and small cycles. Additional operators based on curvature invariants would be controlled, skirting the problem of small cycles.

There are of course other issues that must traditionally be confronted in any Kaluza–Klein scenario, such as the incorporation of chiral fermions and stabilization of moduli. In the meantime, it is encouraging that a braneless cosmos might hide large-volume extra dimensions. While all fields are smeared out over the large-volume, interactions with gauge fields are concentrated over relatively small cycles and thereby manage to remain undiluted. It is intriguing to imagine that we live smeared out over a large drum and our experience of forces other than gravity are an illusion created by the cadence of small hidden subspaces.

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#### References

- [1] N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B 429 (1998) 263, hep-ph/9803315.
- [2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B 436 (1998) 257, hep-ph/9804398.
- [3] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, hep-ph/9905221.
- [4] B. Greene, D. Kabat, J. Levin, D. Thurston, arXiv:1001.1423, 2010.
- [5] N. Kaloper, J. March-Russell, G.D. Starkman, M. Trodden, Phys. Rev. Lett. 85 (2000) 928, hep-ph/0002001.
- [6] G.D. Starkman, D. Stojkovic, M. Trodden, Phys. Rev. D 63 (2001) 103511, hep-th/0012226.
- [7] G.D. Starkman, D. Stojkovic, M. Trodden, Phys. Rev. Lett. 87 (2001) 231303, hep-th/0106143.
- [8] S. Nasri, P.J. Silva, G.D. Starkman, M. Trodden, Phys. Rev. D 66 (2002) 045029, hep-th/0201063.
- [9] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. (1921) 966.
- [10] O. Klein, Z.F. Physik (1926) 895.
- [11] S. Weinberg, Phys. Lett. B 125 (1983) 265.
- [12] Y.M. Cho, P.G.O. Freund, Phys. Rev. D 12 (1975) 1711.
- [13] K.R. Dienes, Phys. Rev. Lett. 88 (2002) 011601, hep-ph/0108115.
- [14] K.R. Dienes, A. Mafi, Phys. Rev. Lett. 88 (2002) 111602, hep-th/0111264.
- [15] E. Witten, Nucl. Phys. B 186 (1981) 412.
- [16] P. Buser, Math. Z. 162 (1978) 87.
- [17] P. Buser, Discrete Appl. Math. 9 (1984) 105.

- [18] A. Selberg, in: A.L. Whiteman (Ed.), *Theory of Numbers*, in: *Proc. Sympos. Pure Math.*, vol. 8, 1965, p. 1.
- [19] W. Luo, Z. Rudnick, P. Sarnak, *Geom. Funct. Anal.* 5 (1995) 387.
- [20] R. Brooks, E. Makover, *Journal D'analyse Mathématique* 83 (2001) 243.
- [21] R. Brooks, E. Makover, *Electronic Research Announcements of the American Mathematical Society* 5 (1999) 76.
- [22] A. Strominger, *Nucl. Phys. B* 451 (1995) 96, hep-th/9504090.
- [23] J. Weeks, <http://www.geometrygames.org/snappea/>.